DMC 2024 Guts Solutions

April 2024

1 Solutions

1. We can ignore the first turn - since the two other planets are symmetric. Thus, there is a $\boxed{1/2}$ chance it returns to the initial planet.

2. $3000 - 976 = 2024$, $2024 = 46 \times 44$ by difference of squares. Thus, $2024/23 = \boxed{88}$.

3. The region bounded one-eight of a unit circle. Thus, the area is $\boxed{\dfrac{\pi}{8}}$.

4. By AM-GM, $x + y + z \geq 3\sqrt[3]{xyz}$. Thus, $8 \geq 3\sqrt[3]{xyz}$, or $xyz \leq \frac{512}{27}$. $512 + 27 = \boxed{539}$.

5. Let the palindrome be *abcba*. Notice that the first digit cannot be 0, and must be 1-9. In these 9 numbers, there is a symmetry between all remainders when divided by 3. Thus, we leave it for last. There are 10×10 ways to choose b and c. However, these two will limit the last choice to have a certain remainder when divided by 3. Because the choices are symmetrical in a, there are always 3 ways. $3 \times 10 \times 10 = \boxed{300}$

6. For this problem, we count backwards. $(((\frac{1}{2} \times 2) + \frac{1}{2}) \times 2 + \frac{1}{2}) \times 2 = \boxed{7}$

7. Notice $45^2 = 2025$. Thus, there are $\boxed{44}$ perfect squares between 1 and 2024 inclusive.

8. Let v be John's walking speed in meters per minute. Bessie then has a walking speed of $(v + 130)$. Setting up an equation, $15v + 10(v + 130) = 2800$, $v = \boxed{60}$.

9. Let x be the side length. $x^2 + 10 = (x + 1)^2 - 15$, $x = 12$. $12^2 + 10 = \boxed{154}$

10. Solve the system of 3 equations to find $x = \frac{37}{12}$, $y = \frac{7}{8}$, $z = -\frac{29}{24}$. $x + y + z = \boxed{\dfrac{11}{4}}$

11. Let x represent the concentration of Solution A. Let y represent the concentration of Solution B. Given that $x = 2y$. Initial alcohol content $= 15\% \times 1000 = 150$. The final mixture's mass $= 1000 + 100 + 400 = 1500$. Final alcohol content $= 14\% \times 1500 = 210$. Alcohol from Solution A $= x\% \times 100grams$. Alcohol from Solution B $= y\% \times 400$. Total alcohol content = Initial alcohol content + Alcohol from A + Alcohol from B. $150 + x/100 \times 100 + y/100 \times 400 = 210$, $x = \boxed{20\%}$.

12. $2016 = 2^5 \times 3^2 \times 7$. To make it a perfect square, the powers must be even: $2 \times 7 = \boxed{14}$.

13. $3 \wedge 4 = 5$. $3 \circ 5 = 15$. $15 \wedge 2 = \boxed{13}$.

14. Notice that ABC is a right triangle with right angle at A. Because of this, $XOZA$ is a square, and $\angle XOZ = 90°$. Since $\angle XYZ$ faces the same chord as XOZ, but is on the circle, $\angle XYZ = \angle XOZ \div 2 = \boxed{45°}$.

15. Move the $4x$, $4y$, $4z$ over to the left hand side and complete the squares:

$$(x-2)^2 + (y-2)^2 + (z-2)^2 = 25$$

Finding the number of ordered triples (x, y, z) satisfying the above is the same as finding the number of ordered triples (a, b, c) satisfying

$$a^2 + b^2 + c^2 = 25$$

Through some trial, we find the only possibilities are permutations of $(\pm 3, \pm 4, 0)$ and $(\pm 5, 0, 0)$. Thus, there are $3 \times 2 \times 2^2 + 3 \times 2 = \boxed{30}$

16. Notice that every term is in the form $\frac{n+1}{n!} = \frac{1}{n!} + \frac{1}{(n-1)!}$. Thus, our sum is the telescoping series $\frac{1}{0!} + \frac{1}{1!} - \frac{1}{1!} - \frac{1}{2!} + \frac{1}{2!} + \frac{1}{3!} - ... - \frac{1}{2023!} - \frac{1}{2024!} = 1 - \frac{1}{2024!}$. $1 + 2024 = \boxed{2025}$.

17. W.L.O.G, suppose the square has vertices $(0,0), (1,0), (0,1), (1,1)$ It is obvious that no centroid can be in the region bounded by $(\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{2}{3})$. All other points can be covered because the height and midpoint of the triangle's base can be adjusted to sweep all of these points. Therefore, our answer is $(1 - (\frac{1}{3})^2) \times 99 = \boxed{88}$.

18. For the laser to return back to its starting point, the sum of the lengths of the arcs must sum to a multiple of 360. Furthermore, every chord after a reflection covers the same distance. Therefore, we express the length of each arc as x, and get the equation $30x = 360k$ or $x = 12k$, where $k \in \mathbb{N}^+$ and $x \in (0, 180)$. Therefore, the possible arc lengths x are $12, 24, 36, \cdots, 168$. To find all possible $\angle BAA'$, we have the expression $\frac{180-x}{2}$, since the central angle or arc length forms a isosceles triangle with the two radii. After this transformation, the sum becomes $6 + 12 + 18 + \cdots + 84 = \boxed{630}$

19. The expected value of the number of heads to appear on any given coin toss is $\frac{1}{2}$. The expected value of the product of the value on two dice is the same as the average roll. We can find this through first finding the total of all the products: $(1+2+3+4+5+6) \times (1+2+3+4+5+6)$, then dividing by 36 to find the average. Thus, we get the expression $\frac{21^2}{36 \times 2} = \boxed{\frac{49}{8}}$. Note that we can reach the same result by $E_{1 \times 2} = E_1 \times E_2$, since the expected values are independent.

20. Let $x = 20242018$. The expression becomes $(x-6)(x-2)(x+2)(x+6) = (x^2-4)(x^2-36) = x^4 - 40x + 144$. Completing the square, $(x^2-20)^2 - 256$. Thus, we must add $\boxed{256}$.

21. The set of points closer to X than any other point are bounded by the perpendicular (plane) bisectors of the radii to each vertex. Doing this, we see that the planes chop off 4 corners of equal volume. We know the centroid ratio in a tetrahedron is $3:1$, so the height of the small tetrahedron is $\frac{3}{8}$ of the large tetrahedron. This means the volume is $\frac{27}{512}$ of the large tetrahedron. Thus, the four corners make up for $\frac{27}{128} \times 1024$, and the remaining volume is $\frac{101}{128} \times 1024 = \boxed{808}$.