DMC 2024 Team General Solutions

April 2024

1. We can set up the equation $\frac{2x}{3} = \frac{x}{3} + 6$. By solving the equation, $x = \boxed{18}$

2. The one's digit of the number to be squared has to be 5 by modular arithmetic. By trial and error, $25^2 = \boxed{625}$

3. $a_1 + a_2 + a_3 = 12$ and $a_1 \leq a_2 \leq a_3$ Using brute force counting running through all possibilities, there are $\boxed{12}$ solutions.

4. Using modular arithmetic, a pattern emerges such that $f_n = 4^n \pm 1$. Therefore, $f_n = \pm 1 \pmod{64}$. For the 100th iteration, the remainder is $-1 = \boxed{63}$

5. Set $x = 1$ and $y = 1$, such that $f(1)^2 = 2f(1)$. Solving the for $f(1)$, $f(1) = 0$ or 2. However, $f(x)$ is nonzero, therefore $f(x)$ must be 2. As the function is distributive $f(2024) = 2024 \times f(1) = \boxed{4048}$.

6. Using casework, only 8 valid cases exists, which can be separated into three categories (D represents a digit and O represents an operation):

   (1) DDDDDD (probability $\frac{9^6}{13^6}$)

   (2) DODDDD (probability $\frac{4\times 9^5}{13^6}$)

   (3) DDODDD (probability $\frac{4\times 9^5}{13^6}$)

   (4) DDDODD (probability $\frac{4\times 9^5}{13^6}$)

   (5) DDDDOD (probability $\frac{4\times 9^5}{13^6}$)

   (6) DODODD (probability $\frac{4^2\times 9^4}{13^6}$)

   (7) DODDOD (probability $\frac{4^2\times 9^4}{13^6}$)

   (8) DDODOD (probability $\frac{4^2\times 9^4}{13^6}$)

   The first case is the first category and has probability $\frac{9^6}{13^6}$. The second to fifth cases belong to the second category and has one operation, totaling probability $4\times\frac{4\times 9^5}{13^6}$. The sixth to eighth cases belong to the third category and has two oeprations, totaling probability $3\times\frac{4^2\times 9^4}{13^6}$. Summing, we get

$$\begin{aligned}
\text{total probability} &= \frac{9^6}{13^6} + 4\times\frac{4\times 9^5}{13^6} + 3\times\frac{4^2\times 9^4}{13^6} \\
&= \frac{3^{12}}{13^6} + \frac{4^2\times 3^{10}}{13^6} + \frac{4^2\times 3^9}{13^6} \\
&= 3^9\times\frac{3^3}{13^6} + \frac{4^2\times 3}{13^6} + \frac{4^2}{13^6} \\
&= 3^9\times\frac{3^3+4^2\times 3+4^2}{13^6} \\
&= 3^9\times\frac{27+48+16}{13^6} \\
&= 3^9\times\frac{91}{13^6} \\
&= 3^9\times\frac{7\times 13}{13^6} \\
&= \frac{3^9\times 7}{13^5}
\end{aligned}$$

   Since the problem asks for $(p+q+r)\times(a+b+c)$ when expressed as $\frac{p^aq^b}{r^c}$, we get $(3+7+13)\times(9+1+5) = 23\times 15 = \boxed{345}$

7. Let $\overline{AD} = \overline{BC} = x$. Then, $\overline{BX} = x + 15$ and $\overline{AY} = x + 8$. Since $\overline{AY}$ and $\overline{AX}$ are both radii, $\overline{AY} = \overline{AX}$. Notice ΔXAB is an isosceles right triangle. This implies that $\sqrt{2} \times \overline{AX} = \overline{BX}$. Substituting, we find $\sqrt{2}(x+8) = x + 15$. Solving this equation, $x = \frac{15-8\sqrt{2}}{\sqrt{2}-1} = 7\sqrt{2} - 1$. Thus, the radius is $7\sqrt{2}+7$, and the area is $\pi(147+98\sqrt{2})$. $147+98+2 = \boxed{247}$.

8. By vieta's theorem, $\frac{a+b}{c} + \frac{b+c}{a} = 574$, $\frac{a+b}{c} \times \frac{b+c}{a} = 17$. Let $k = \frac{a+c}{b}$. Adding k to the first equation and multiplying it to the second equation, we get:

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{a+c}{b} = \frac{a^2b + a^2c + b^2a + b^2c + c^2a + c^2b}{abc} = 574 + k$$
$$\frac{a+b}{c} \times \frac{b+c}{a} \times \frac{a+c}{b} = \frac{2abc + a^2b + a^2c + b^2a + b^2c + c^2a + c^2b}{abc} = 17k$$

Noticing that the $2abc$ is the only difference between the two equations, we find that $574 + k = 17k - 2$, and $k = \boxed{36}$.