DMC 2024 Team General Round (240 points)
April 27th, 2024

1. (23 points) In Mr. Reid's class, the students want to watch a movie. However, they cannot decide which movie to watch. $\frac{1}{3}$ of the students want to watch Oppenheimer, but $\frac{2}{3}$ want to watch Barbie. 6 more students want to watch Barbie than Oppenheimer. How many students are in Mr. Reid's class?

2. (25 points) Zhang is thinking of a positive integer with 3 distinct digits. Given that its unit digit is 5 and that it can be expressed as the square of an integer, what number is Zhang thinking of?

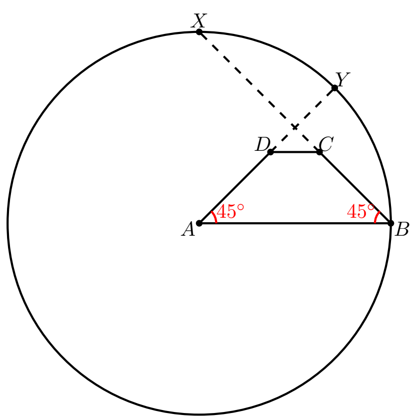
3. (27 points) In preparation for the DMC, a student decides to solve 12 problems over the course of 3 days. How many ways can they do this, provided that they solve at least 1 problem a day, and solve at least as many problems as the previous day?

4. (29 points) Let $f_n = 3f_{n-1} + 4f_{n-2}$ and $f_1 = 2, f_2 = 3$. Find the remainder when f_{100} is divided by 64.

5. (31 points) Suppose a nonzero (not equal to 0 for all inputs) function f satisfies $f(x + y) = f(x) + f(y)$ for all real x and y. Suppose $[f(1)]^2 = f(2)$. Find $f(2024)$.

6. (33 points) A calculator has 14 buttons: digits ranging from 1 to 9, the four basic operations (addition, subtraction, multiplication, and division), and the equal sign. A person presses six random buttons that are not the equal sign, and for his seventh button, presses the equal sign. The probability that the seven buttons are pressed with correct syntax can be written in the form of $\frac{p^a q^b}{r^c}$ where p, q, and r are prime numbers and a, b, and c are positive integers. ($23 + 456 =$ and $123456 =$ are valid, but $+12 - 54 =$, $32 + -25 =$, and $-23 + 13 =$ are not). Find $(p + q + r) \times (a + b + c)$.

7. (35 points) Trapezoid $ABCD$ has $\angle A = \angle B = 45°$. Let ω be the circle with radius $\overline{AB}$ centered at A. Extend $\overline{BC}$ through C and $\overline{AD}$ through D to intersect ω at points X and Y, respectively. Given $\overline{DY} = 8$ and $\overline{CX} = 15$, the area of ω can be expressed in the form $\pi(a + b\sqrt{c})$, where a and b are integers and c is an integer that is not a multiple of any square number. Find $a + b + c$.



8. (37 points) For real numbers a, b, and c, suppose that $\frac{a+b}{c}$ and $\frac{b+c}{a}$ are the roots of the polynomial $x^2 - 574x + 17$. Find $\frac{a+c}{b}$.