DMC 2024 Individual Relay Round (240 points)
April 27th, 2024

## Set Alpha

1. (12 point) Suppose the following is true of the asexual sea star population in the country of Georgia:

   - 10% of sea stars produce exactly 0 offspring.
   - 20% of sea stars produce exactly 1 offspring.
   - 10% of sea stars produce exactly 2 offspring.
   - 30% of sea stars produce exactly 3 offspring.
   - 30% of sea stars produce exactly 4 offspring.
   - No sea stars produce more than 4 offspring.

   Let $\alpha_1$ be the average number of offspring that a Georgian sea star releases. What is $\alpha_1$? Leave $\alpha_1$ as a fraction or demical. Do not round.

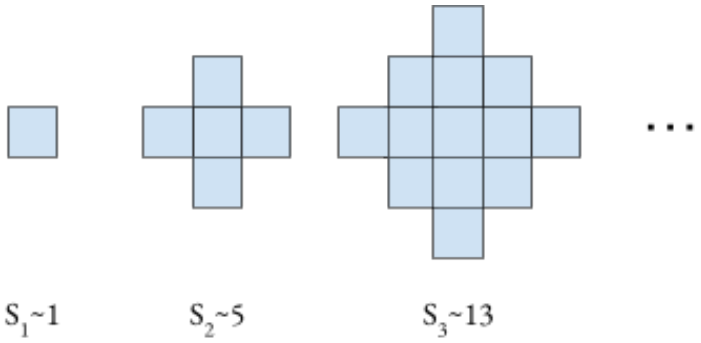
2. (13 points) The current sea star population in Georgia is 1 million. The average number of offspring produced by a Georgian sea star is $\alpha_1$. Assume that every sea star offspring matures into a grown sea star and that within the 1-year lifespan of every sea star, it releases offspring and dies of natural causes. Let $\alpha_2$ million be the expected sea star population in two years. What is $\alpha_2$?

3. (14 points) Let $\eta$ be the greatest integer that is less than or equal to $\alpha_2$. Suppose $\eta$ students from Tbilisi Seaside Elementary are watching a play together. Two students, Nino and Elene, wish to be seated together. Let $\alpha_3$ be the number of ways that the $\eta$ students can be lined up in a single row such that Nino and Elene are next to each other. What is $\alpha_3$?

4. (15 points) After learning about prime factorization in math class at Seaside, one student found the numbers $x$, $y$, and $z$ to be the prime factors of $\alpha_3$ that satisfy the inequality $x < y < z$. Let $x^3 \cdot y^2 \cdot z^1 = \varphi$, and let $\alpha_4 = \varphi - \alpha_3$. What is $\alpha_4$?

## Set Beta

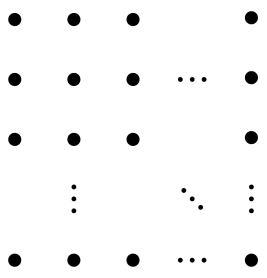
5. (13 points) Following the pattern shown in the graphic below, we construct a sequence of shapes, each consisting of small squares. $S_1$, $S_2$, and $S_3$ are formed using 1, 5, and 13 squares respectively. Let the number of small squares forming $S_6$ be $\beta_1$. What is $\beta_1$?




6. (14 points) Let $\beta_1$ be the side length of both an equilateral triangle and a square. The ratio of the area of the triangle to the area of the square is of the form $\frac{\sqrt{p}}{q}$, where p is not a multiple of any square number. Let the sum $p + q = \beta_2$. What is $\beta_2$?

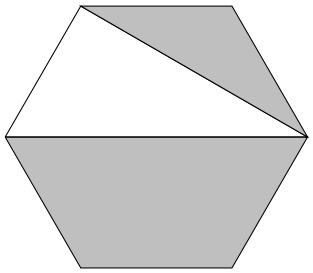
7. (15 points) Peter has a jar containing $\beta_2$ pickles. He borrows two more pickles from his friend Piper. If Peter has an even number of pickles at the start of the day, he eats half the pickles. If Peter has an odd number of pickles at the start of the day, he borrows a pickle from Piper to make the total an even number and then eats half. Peter continues this process until he has one pickle left. At this point, he owes Piper $\beta_3$ pickles. What is $\beta_3$?

8. (16 points) How many rectangles, $\beta_4$, exist whose vertices are the lattice points of a $\beta_3 \times \beta_3$ grid, and whose sides are either vertical or horizontal?

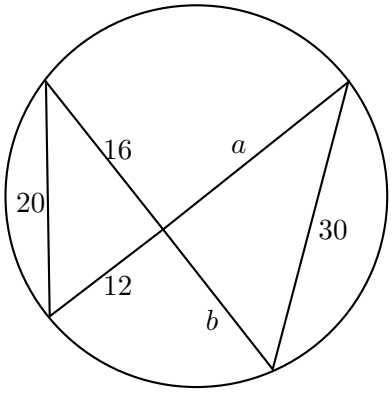


## Set Gamma

9. (14 points) Kevin owns a regular hexagonal dartboard divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. Let $\gamma_1$ be the probability that the dart lands within the shaded region. Find $\gamma_1$ as a reduced fraction.



10. (15 points) In his free time, Eric decided to design a bow tie inscribed in a circle as shown. However, he only remembers that the area of the triangle on the left is $(\gamma_1)^2$ times the area of the triangle on the right. Find $\gamma_2 = a + b$, where $a$ and $b$ are the remaining measurements.




11. (16 points) Let $\gamma_3$ be the units digit of the sum $1^{\gamma_2} + 2^{\gamma_2} + 3^{\gamma_2} + 4^{\gamma_2} + 5^{\gamma_2} + 6^{\gamma_2} + 7^{\gamma_2} + 8^{\gamma_2} + 9^{\gamma_2} + 10^{\gamma_2}$. What is $\gamma_3$?

12. (17 points) Let $i^2 = -1$. The expression

$$\left(\frac{1-2i}{3+4i} - \frac{2+i}{5i}\right)^{\gamma_3}.$$

can be expressed as $-\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. What is $\gamma_4 = m + n$?

## Set Delta

13. (15 points) Find the smallest four digit number $\delta_1$ such that when an army of $\delta_1$ people are configured into a square formation, the formation is missing one soldier in the last row. When the same army is arranged into rows of 3 or rows of 5, the formation is still missing one soldier in the last row.

14. (16 points) An army with $\delta_1 + 1$ people needs to cross a river. They have 5 ships, and they divide themselves into the ships randomly. Of the $\delta_1 + 1$ soldiers, two soldiers, Daniel and Eric, are friends. If the probability that Daniel and Eric are put in the same ship is $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers, and $\delta_2 = a + b$, what is the value of $\delta_2$?

15. (17 points) $10 \times \delta_2$ soldiers are eating dinner. All the soldiers eating dinner were born in the same year, and the soldiers' birthdays follow an interesting pattern - they are configured in such a way that there are a minimum number of soldiers with the same birth week. How many ways, $\delta_3$, are there to select a group of 4 soldiers out of the $10 \times \delta_2$ soldiers such that there are only two unique birth weeks in the group? (Assume that there are only 52 weeks in a year).

16. (18 points) $\delta_3$ soldiers are sitting around a circular table. Each of these $\delta_3$ soldiers have a mutual best friend such that there are $\frac{\delta_3}{2}$ pairs of best friends. Each person is sitting across from their best friend. Assuming all rotations are the same. The number of ways to arrange the $\delta_3$ people can be represented as $k! \times 2^j$, where $k$ and $j$ are integers. Find $k + j = \delta_4$.